Kustin–Miller unprojection without complexes

Stavros Papadakis* Miles Reid†

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Abstract

Gorenstein projection plays a key role in birational geometry; the typical example is the linear projection of a del Pezzo surface of degree d to one of degree $d-1$, but variations on the same idea provide many of the classical and modern birational links between Fano 3-folds. The inverse operation is the Kustin–Miller unprojection theorem, which constructs "more complicated" Gorenstein rings starting from "less complicated" ones (increasing the codimension by 1). We give a clean statement and proof of their theorem, using the adjunction formula for the dualising sheaf in place of their complexes and Buchsbaum–Eisenbud exactness criterion. Our methods are scheme theoretic and work without any mention of the ambient space. They are thus not restricted to the local situation, and are well adapted to generalisations.

Section 2 contains examples, and discusses briefly the applications to graded rings and birational geometry that motivate this study; see also Papadakis [P1] and Reid [R3]–[R4].

1 The theorem

Let $X = \operatorname{Spec} \mathcal{O}_X$ be a Gorenstein local scheme and $I \subset \mathcal{O}_X$ an ideal sheaf defining a subscheme $D = V(I) \subset X$ that is also Gorenstein and has codimension 1 in X. We assume that all schemes are Noetherian. We do not assume anything else about the singularities of X and D, although an important case in applications is when X is normal and D a Weil divisor.

The adjunction formula (compare Reid [R], Appendix to Section 2) gives

$$\omega_D = \mathcal{E}xt^1(\mathcal{O}_D, \omega_X).$$

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To calculate the $\mathcal{E}xt$, we apply the derived functor of $\mathcal{H}om$ to the exact sequence $0 \to I \to \mathcal{O}_X \to \mathcal{O}_D \to 0$ into ω_X, obtaining the usual adjunction exact sequence

$$0 \to \omega_X \to \mathcal{H}om(I, \omega_X) \xrightarrow{\mathrm{res}_D} \omega_D \to 0,$$

where res_D is the residue map. For example, in the case that X is normal and D a divisor, the second map is the standard Poincaré residue map $\mathcal{O}_X(K_X + D) \to \mathcal{O}_D(K_D)$.

Lemma 1.1 *The $\mathcal{O}_X$ module $\mathcal{H}om(I, \omega_X)$ is generated by two elements i and s, where i is a basis of ω_X and $s \in \mathcal{H}om(I, \omega_X)$ satisfies*

(i) $s \colon I \hookrightarrow \omega_X$ *is injective;*

(ii) $\overline{s} = \mathrm{res}_D(s)$ *is a basis of* ω_D.

Proof Choose bases $i \in \omega_X$, $\overline{s} \in \omega_D$ and any lift $s \mapsto \overline{s}$. Then everything holds except (i). We achieve (i) by a simple exercise in primary decomposition: write $X_i \subset X$ for the reduced irreducible components and P_i for the corresponding minimal prime ideals, so that $X_i = V(P_i)$. Then $\operatorname{Ass} \mathcal{O}_X = \{P_i\}$ because X is Cohen–Macaulay.

We have $\ker s \subset I$, so its Ass also consists of irreducible components, namely those on which s vanishes. Choose $f \in \mathcal{O}_X$ such that

$$\begin{cases} f \notin P_i & \text{for } P_i \in \operatorname{Ass}(\ker s), \\ f \in P_i & \text{for } P_i \notin \operatorname{Ass}(\ker s); \end{cases}$$

in other words, f is nonzero (thus generically a unit) on each component where s vanishes, and f vanishes along every component at which s is a unit. Now replacing s by $s + fi$ gives (i). QED

We view s as defining an isomorphism $I \to J$, where $J \subset \omega_X = \mathcal{O}_X$ is another ideal. Choose a set of generators $f_1, \dots, f_k$ of I and write $s(f_i) = g_i$ for the corresponding generators of J. We view $s = g_i/f_i$ as a rational function having I as ideal of denominators and J as ideal of numerators (compare Remark 1.3). Unprojection is simply the graph of s.

Definition 1.2 Let S be an indeterminate. The *unprojection ring* of D in X is the ring $\mathcal{O}_X[s] = \mathcal{O}_X[S]/(Sf_i - g_i)$; the *unprojection* of D in X is its Spec, that is,

$$Y = \operatorname{Spec} \mathcal{O}_X[s].$$

Clearly, Y is simply the subscheme of $\operatorname{Spec} \mathcal{O}_X[S] = \mathbb{A}^1_X$ defined by the ideal $(Sf_i - g_i)$. Usually Y is no longer local (see Example 2.2).

Remark 1.3 (1) Clearly $J = \mathcal{O}_X$ if and only if I is principal; if $I = (f)$ then $\mathcal{O}_X[s] = \mathcal{O}_X[1/f]$. We exclude this case in what follows.

(2) We only choose generators for ease of notation here. The ideal defining Y could be written $\big(Sf - s(f) \bigm| f \in I\big)$.

The construction is independent of s: the only choice in Lemma 1.1 is $s \mapsto us + hi$ with $u, h \in \mathcal{O}_X$ and u a unit, which just gives the affine linear coordinate change $S \mapsto uS + h$ in $\mathbb{A}^1_X$.

(3) The total ring of fractions $K(X)$ is defined as $S^{-1}\mathcal{O}_X$ where S is the set of non-zerodivisors, that is, the complement of the associated primes $P_i \in \operatorname{Ass} \mathcal{O}_X$. Then $s\colon I \to J$ is multiplication by an invertible rational function in $K(X)$. For I contains a regular element w (in fact $\operatorname{depth} I = \operatorname{codim} D = 1$, by Matsumura [M], Theorem 17.4), and $s(w)/w \in K(X)$ is independent of the choice of w, because

$$0 = s(w_1w_2 - w_2w_1) = w_1s(w_2) - w_2s(w_1) \quad \text{for } w_1, w_2 \in I.$$

(4) We defined $\mathcal{O}_X[s]$ by generators and relations in Definition 1.2. If X is normal, it equals the subring of $K(X)$ generated by $\mathcal{O}_X$ and s. Sketch proof: The point is to prove "only linear relations", that is, any relation of the form $as^2 + bs + c = 0$ (etc.) is in the ideal generated by the linear relations $sf_i - g_i$; this is clear, because if $(as+b)s = -c \in \mathcal{O}_X$ then $as + b = -c/s$ cannot have any divisor of poles.

Lemma 1.4 *Write $N = V(J) \subset X$ for the subscheme with $\mathcal{O}_N = \mathcal{O}_X/J$.*

(a) No component of X is contained in N.

(b) Every associated prime of $\mathcal{O}_N$ has codimension 1.

If X is normal then D and N are both divisors, with $\operatorname{div} s = N - D$. More generally, set $n = \dim X$; then (a) says that $\dim N \le n - 1$, and (b) says that $\dim N = n - 1$ (and has no embedded primes).

Proof As we have just said, I contains a regular element $w \in \mathcal{O}_X$. Then $v = s(w) \in J$ is again regular (obvious), and (a) follows.

(b) follows from (a) and Step 2 of the proof of Theorem 1.5: for (1.3) below gives $n - 1 \le \operatorname{depth} \mathcal{O}_X/J < n = \dim X$.

For a direct proof of (b), note first that $vI = wJ$. We prove that every element of $\operatorname{Ass}(\mathcal{O}_X/vI) = \operatorname{Ass}(\mathcal{O}_X/wJ)$ is a codimension 1 prime; the lemma follows, since $\operatorname{Ass}(\mathcal{O}_X/J) = \operatorname{Ass}(w\mathcal{O}_X/wJ) \subset \operatorname{Ass}(\mathcal{O}_X/wJ)$. Clearly,

$$\operatorname{Ass}(\mathcal{O}_X/vI) \;\subset\; \operatorname{Ass}(\mathcal{O}_X/I) \cup \operatorname{Ass}(I/vI).$$

For any $P \in \operatorname{Ass}(I/vI)$, choose $x \in I$ with $P = (vI : x) = \operatorname{Ann}(\overline{x} \in I/vI)$. One sees that

$$\begin{cases} x \in \mathcal{O}_X v \implies P \in \operatorname{Ass}(\mathcal{O}_X/I), \\ x \notin \mathcal{O}_X v \implies P \subset Q \text{ for some } Q \in \operatorname{Ass}(\mathcal{O}_X/v\mathcal{O}_X). \end{cases}$$

Since every associated prime of $\mathcal{O}_X/v\mathcal{O}_X$ has codimension 1, this gives

$$\operatorname{Ass}(\mathcal{O}_X/vI) \;\subset\; \operatorname{Ass}(\mathcal{O}_X/I) \cup \operatorname{Ass}(\mathcal{O}_X/v\mathcal{O}_X). \quad \text{QED}$$

Theorem 1.5 (Kustin and Miller [KM]) *The element $s \in \mathcal{O}_X[s]$ is a non-zerodivisor, and $\mathcal{O}_X[s]$ is a Gorenstein ring.*

Step 1 We first prove that

$$S\mathcal{O}_X[S] \cap (Sf_i - g_i) = S(Sf_i - g_i), \tag{1.1}$$

under the assumption that $s\colon I \to J$ is an isomorphism.

For suppose $b_i \in \mathcal{O}_X[S]$ are such that $\sum b_i(Sf_i - g_i)$ has no constant term. Write b_{i0} for the constant term in b_i, so that $b_i - b_{i0} = Sb'_i$. Then $\sum b_{i0} g_i = 0$. Since $s\colon f_i \mapsto g_i$ is injective, also $\sum b_{i0} f_i = 0$. Thus the constant terms in the b_i don't contribute to the sum $\sum b_i(Sf_i - g_i)$, which proves (1.1).

The natural projection $\mathcal{O}_X[S] \twoheadrightarrow \mathcal{O}_X$ takes $(Sf_i - g_i) \twoheadrightarrow J = (g_i)$, and (1.1) calculates the kernel. This gives the following exact diagram:

$$\begin{array}{ccccccccc}
0 & \to & (Sf_i - g_i) & \xrightarrow{S} & (Sf_i - g_i) & \to & J & \to & 0 \\
 & & \bigcap & & \bigcap & & \bigcap & & \\
0 & \to & \mathcal{O}_X[S] & \xrightarrow{S} & \mathcal{O}_X[S] & \to & \mathcal{O}_X & \to & 0 \\
 & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \\
 & & \mathcal{O}_X[s] & \xrightarrow{s} & \mathcal{O}_X[s] & \to & \mathcal{O}_X/J & \to & 0
\end{array}$$

The first part of the theorem follows by the Snake Lemma.

Step 2 To prove that N is Cohen–Macaulay, recall that

$$\operatorname{depth} M = \inf \left\{ i \geq 0 \;\middle|\; \operatorname{Ext}^i_{\mathcal{O}_X}(k, M) \neq 0 \right\}$$

for M a finite $\mathcal{O}_X$-module over a local ring $\mathcal{O}_X$ with residue field $k = \mathcal{O}_X/\mathfrak{m}$ (see [M], Theorem 16.7). We have two exact sequences

$$\begin{gathered} 0 \to I \to \mathcal{O}_X \to \mathcal{O}_X/I \to 0 \\ 0 \to J \to \mathcal{O}_X \to \mathcal{O}_X/J \to 0. \end{gathered} \tag{1.2}$$

By assumption, $\mathcal{O}_X$ and $\mathcal{O}_X/I$ are Cohen–Macaulay, therefore

$$\begin{aligned} &\operatorname{Ext}^i_{\mathcal{O}_X}(k, \mathcal{O}_X) = 0 \quad \text{ for } 0 \leq i < n \\ \text{and} \quad &\operatorname{Ext}^i_{\mathcal{O}_X}(k, \mathcal{O}_X/I) = 0 \quad \text{ for } 0 \leq i < n-1, \end{aligned}$$

where $n = \dim X$. Thus

$$\operatorname{Ext}^i_{\mathcal{O}_X}(k, I) = 0 \quad \text{ for } 0 \leq i < n, \tag{1.3}$$

and the Ext long exact sequence of (1.2) gives also

$$\operatorname{Ext}^i_{\mathcal{O}_X}(k, \mathcal{O}_X/J) = 0 \quad \text{ for } 0 \leq i < n-1.$$

Therefore $\mathcal{O}_N = \mathcal{O}_X/J$ is Cohen–Macaulay.

Step 3 We prove that $\omega_N \cong \mathcal{O}_N$ by running the argument of Lemma 1.1 in reverse. Recall that $\mathcal{H}om(I, \omega_X)$ is generated by two elements i, s, where i is a given basis element of ω_X viewed as a submodule $\omega_X \subset \mathcal{H}om(I, \omega_X)$, and s is our isomorphism $I \to J \subset \omega_X$.

We write j for the same basis element of ω_X viewed as a submodule of $\mathcal{H}om(J, \omega_X)$, and $t = s^{-1}\colon J \to I \subset \omega_X$ for the inverse isomorphism. Now $s\colon I \to J$ induces a dual isomorphism

$$s^*\colon \mathcal{H}om(J, \omega_X) \to \mathcal{H}om(I, \omega_X),$$

which is defined by $s^*(\varphi)(v) = \varphi(s(v))$ for $\varphi\colon J \to \omega_X$. By our definitions, clearly $s^*(j) = s$ and $s^*(t) = i$. Since s^* is an isomorphism, it follows that $\mathcal{H}om(J, \omega_X)$ is generated by t and j. Therefore the adjunction exact sequence

$$0 \to \omega_X \to \mathcal{H}om(J, \omega_X) \to \omega_N \to 0$$

gives $\omega_N = \mathcal{O}_N \bar{t}$. This completes the proof that $\mathcal{O}_N$ is Gorenstein.

Alternative proof of Step 3 We worked out the above slick proof of Step 3 by untangling the following essentially equivalent argument, which may be more to the taste of some readers.

We set up the following exact commutative diagram:

$$\begin{array}{ccccccccc}
0 & \to & I & \xrightarrow{s} & \mathcal{O}_X & \to & \mathcal{O}_N & \to & 0 \\
 & & \cap & & \cap & & & & \\
0 & \to & \mathcal{O}_X & \xrightarrow{s_2} & \mathcal{H}om(I, \omega_X) & \to & L & \to & 0 \\
 & & \twoheaddownarrow & & \twoheaddownarrow & & & & \\
 & & \mathcal{O}_D & \xrightarrow{s_3} & \omega_D & & & &
\end{array}$$

The first column is just the definition of $\mathcal{O}_D$. The second column is the identification of $\mathcal{O}_X$ with ω_X composed with the adjunction formula for ω_D.

The first row is the multiplication $s\colon I \to J$ composed with the definition of $\mathcal{O}_N$. To make the first square commute, the map s_2 must be defined by

$$s_2(a)(b) = s(ab) \quad \text{for } a \in \mathcal{O}_X \text{ and } b \in I. \tag{1.4}$$

We identify its cokernel L below. The first two rows induce the map s_3. Since s_2 takes $1 \in \mathcal{O}_X$ to $s \in \mathcal{H}om(I, \omega_X)$, it follows that s_3 takes $1 \in \mathcal{O}_D$ to $\overline{s} \in \omega_D$ as in Lemma 1.1, and therefore s_3 is an isomorphism.

Now the second row is naturally identified with the adjunction sequence

$$0 \to \omega_X \to \mathcal{H}om(J, \omega_X) \to \omega_N \to 0.$$

The point is just that $s\colon I \cong J$, and s_2 is the composite

$$0 \to \omega_X \hookrightarrow \mathcal{H}om(I, \omega_X) \xrightarrow{s^*} \mathcal{H}om(J, \omega_X),$$

by its definition in (1.4). The Snake Lemma now gives $\mathcal{O}_N \cong L = \omega_N$. Therefore, as before, N is Gorenstein.

In what follows, we prove that $\mathcal{O}_X[s]$ is Gorenstein: that is (see [M], Definition on p. 145 and Theorem 18.2), its localisation $(\mathcal{O}_X[s])_\mathfrak{n}$ at any maximal ideal $\mathfrak{n}$ of $\mathcal{O}_X[s]$ is Gorenstein. Since $\mathfrak{n} \cap \mathcal{O}_X = \mathfrak{p} \in \operatorname{Spec} \mathcal{O}_X$, and localising $\mathcal{O}_X$ at $\mathfrak{p}$ preserves all the assumptions, we need only to consider $\mathfrak{n}$ lying over $\mathfrak{n} \cap \mathcal{O}_X = \mathfrak{m}$.

Step 4 – special case If $s \in \mathfrak{n}$ we are done by Step 3: s is a regular element by the first part of the theorem and $(\mathcal{O}_X[s])_{\mathfrak{n}}/(s) = (\mathcal{O}_N)_{\mathfrak{n}}$ is Gorenstein by Step 3. This argument gives nothing if $s \notin \mathfrak{n}$: s is a unit and the quotient by (s) is zero. (This was a small gap in the Nov 2000 preprint of this paper.)

The same argument works if $s - a \in \mathfrak{n}$ for some $a \in \mathcal{O}_X$: we can arrange that $s - a \colon I \to \mathcal{O}_X$ is injective (wiggle if necessary by an element of $\mathfrak{m}$, as in Lemma 1.1, (i)), then replace $s \mapsto s - a$ in the construction by a coordinate change as in Remark 1.3, (2). If $\mathcal{O}_X$ contains an algebraically closed field k that maps isomorphically to the residue field $\mathcal{O}_X/\mathfrak{m}$ (the main case in many applications), this completes the proof.

Step 5 – general case We use an extension of the residue field $k = \mathcal{O}_X/\mathfrak{m}$ to reduce the general case to the case $s \in \mathfrak{n}$. We need two facts.

Exercise 1.6 Let $(A, \mathfrak{m}, k)$ be a local ring, and $k \subset L$ a finite extension of the residue field. Then there exists an extension ring $A \subset B$ such that

(i) $B \cong A^{\oplus N}$ is a free A-module of rank $N = [L : k]$;

(ii) B is local with maximal ideal $\mathfrak{m}' = \mathfrak{m} \cdot B$ and $B/\mathfrak{m}' = L$.

[Hint: Do a primitive extension $k \subset k_1$ first, then induction on $[L : k]$.]

Proposition 1.7 *Let $(A, \mathfrak{m}, k)$ be a local ring and $A \subset B$ an extension ring that is a finite free A-module. Thus $B \cong A^{\oplus N}$ and $B/\mathfrak{m}B \cong k^{\oplus N}$. In particular, B is semilocal, and its finitely many localisations $(B_i, \mathfrak{n}_i)$ also have quotient rings $B_i/\mathfrak{m}B_i$ that are finite dimensional k-vector spaces.*

(i) $\operatorname{depth} A = \operatorname{depth}_{B_i} B_i$ *for each i; in particular, A is Cohen–Macaulay $\iff$ B is Cohen–Macaulay.*

(ii) B Gorenstein $\implies$ A Gorenstein. (In fact, it is enough that one localisation B_i is Gorenstein.)

(iii) Let $A \subset B$ be as in 1.6, (i–ii). Then A Gorenstein $\implies$ B Gorenstein.

Proof This is a standard result in commutative algebra; see for example Bruns and Herzog [BH], Theorem 1.2.16, p. 13, or Watanabe, Ishikawa, Tachibana and Otsuka [WITO], or Matsumura [M], Theorem 23.4. (We thank John Moody for explaining the argument of [WITO] to us.)

We sketch a direct proof to avoid these references. In (i), an A-regular sequence remains B_i-regular, so $\operatorname{depth} A \le \operatorname{depth} B_i$ is clear. The other way round, $\operatorname{depth} A = 0$ means that $\mathfrak{m} \in \operatorname{Ass} A$, that is, A contains a copy of $A/\mathfrak{m}$. Then, by flatness, each B_i contains a copy of $B_i/\mathfrak{m}B_i$. This is an Artinian local ring, so $\{\mathfrak{n}_i\} = \operatorname{Ass} B_i/\mathfrak{m}B_i \subset \operatorname{Ass} B_i$, and therefore $\operatorname{depth}_{B_i} B_i = 0$. Thus $\operatorname{depth} B_i > 0$ implies also $\operatorname{depth} A > 0$. (i) follows by the usual induction on $\operatorname{depth} A$.

After dividing out by a maximal A-regular sequence $x_1, \dots, x_n \in \mathfrak{m}$, in (ii) and (iii) we can assume that A and B_i are Artinian, and everything comes down to estimates on the dimension of socles. For (ii), suppose that $(A/\mathfrak{m})^{\oplus a} \subset \operatorname{socle} A$. As above, $(B_i/\mathfrak{m}B_i)^{\oplus a} \subset B_i$ by flatness, and $B_i/\mathfrak{m}B_i$ contains at least one copy of $B_i/\mathfrak{n}_i$. Thus $(B_i/\mathfrak{n}_i)^{\oplus a} \subset \operatorname{socle} B_i$. Thus B_i Gorenstein implies $a = 1$ and A is Gorenstein. For (iii), $\dim \operatorname{Hom}_A(k, A) = 1$ and $B \cong A^{\oplus N}$ gives $\dim \operatorname{Hom}_A(k, B) = N$ with $N = [L : k]$. On the other hand, if $L^{\oplus a} \subset \operatorname{socle} B$ then $\dim \operatorname{Hom}_A(k, B) \ge a[L : k]$; therefore $a \le 1$, and B is Gorenstein. QED

We return to the proof of Theorem 1.5. The maximal ideal of the polynomial ring $\mathcal{O}_X[S]$ lying over $\mathfrak{n} \subset \mathcal{O}_X[s]$ is of the form $(\mathfrak{m}, F)$ for some monic polynomial $F \in \mathcal{O}_X[S]$ whose reduction $\overline{F} \in k[S]$ remains irreducible over k. Write $k \subset L$ for a splitting field of $\overline{F}$, so that $\overline{F} = \prod_i (S - \alpha_i)$ with $\alpha_i \in L$, and repeated factors if $\overline{F}$ is inseparable. Let $\mathcal{O}_X \subset \mathcal{O}'$ be a ring extension as in 1.6, (i–ii). We write $I' = I\mathcal{O}'$ and $J' = J\mathcal{O}'$, and extend s to an isomorphism $I' \cong J'$ given by the same formula $f_i \mapsto g_i$. The extended rings $\mathcal{O}'$ and $\mathcal{O}'/I'$ are Gorenstein by Proposition 1.7, (iii). Thus

$$\mathcal{O}'[s] = \mathcal{O}'[S]/(f_i S - g_i) = \mathcal{O}_X[s] \otimes \mathcal{O}'$$

is the unprojection ring constructed from the local ring $\mathcal{O}'$ and the ideal I'.

Now every maximal ideal of $\mathcal{O}'[s]$ that contains $\mathfrak{n}$ is of the form $\mathfrak{n}'_i = (\mathfrak{m}, s - a_i)\mathcal{O}'$, and has residue field L, where $a_i \in \mathcal{O}'$ reduces to $\alpha_i \in L$. Thus each localisation $(\mathcal{O}'[s])_{\mathfrak{n}'_i}$ is Gorenstein by the argument of Step 4, and Proposition 1.7, (ii) gives $(\mathcal{O}_X[s])_{\mathfrak{n}}$ Gorenstein.

This completes the proof of Theorem 1.5. QED

Kustin and Miller's argument

We paraphrase the argument of [KM] for completeness. In addition to our usual assumptions, suppose that everything is contained in an ambient local scheme A, with $\mathcal{O}_X$ and $\mathcal{O}_D$ of finite projective dimension over $\mathcal{O}_A$ (for example, because A is regular). Suppose that $\operatorname{codim}_A X = d$ and

$\operatorname{codim}_A D = d+1$. We write out free resolutions $L_\bullet \to \mathcal{O}_X$ and $M_\bullet \to \mathcal{O}_D$ over $\mathcal{O}_A$. Then the usual properties of resolutions give a map of complexes

$$\begin{array}{ccccccccc} & & 0 \to & L_d & \cdots & L_1 & \to & \mathcal{O}_A & \to & \mathcal{O}_X \\ & & & \downarrow & & \downarrow & & \| & & \downarrow \\ 0 \to M_{d+1} & \to & & M_d & \cdots & M_1 & \to & \mathcal{O}_A & \to & \mathcal{O}_D \end{array} \tag{1.5}$$

Suppose that the ideal of D in $\mathcal{O}_A$ is generated by k elements $f_1, \dots, f_k$, so that $M_1 = k\mathcal{O}_A$.

We identify $L_d = M_{d+1} = \mathcal{O}_A$ and $M_d = M_1^\vee = k\mathcal{O}_A$ by Gorenstein symmetry. Then the tail-end of the complexes gives

$$\begin{array}{rcl} & L_d = \mathcal{O}_A & \to \cdots \\ & \downarrow \ (g_1, \dots, g_k) & \\ 0 \to M_{d+1} = \mathcal{O}_A \xrightarrow{f_1, \dots, f_k} & M_d = k\mathcal{O}_A & \to \cdots \end{array} \tag{1.6}$$

As we have done, Kustin and Miller introduce a new indeterminate S, and write out new equations $Sf_i = g_i$. This gives a new ambient space $\mathbb{A}^1_A = \operatorname{Spec} \mathcal{O}_A[S]$ and a new ideal $I_Y = (I_X, Sf_i - g_i)$. They introduce a new complex by glueing together $L_\bullet \otimes \mathcal{O}_A[S]$ and $M_\bullet \otimes \mathcal{O}_A[S]$, and prove it is a resolution of I_Y by arguments based on the Buchsbaum–Eisenbud criterion [BE].

To check that their construction is the same as ours, take the dual of (1.6), note that $\omega_X = \operatorname{coker}\{L_{d-1}^\vee \to L_d^\vee\}$, and identify the ideal of D in A with $\operatorname{coker}\{(f_i)\colon M_d^\vee \to M_{d+1}^\vee\}$. One shows that the dual diagram induces a map $f_i \mapsto g_i$ from $(f_i) = I_{X,D} \subset \mathcal{O}_X$ to ω_X that provides the second generator of $\mathcal{H}om(I, \omega_X)$ as in Lemma 1.1. See [P1], Section 3 for details.

The advantage of their method is that it gives in theory the complex resolving the new ideal. On the other hand, while it is trivial to say that the map of complexes (1.5) exists, it is hard to calculate, except in the simplest examples (compare 2.8 below); some cases are worked out in Papadakis [P]–[P1] and [R3]. Our construction identifies the final and most important vertical map in (1.5) as a Poincaré residue, and thus as a "determinant", which we can hope to calculate birationally without knowing the finer details of the "matrix" that gave rise to it. (From a philosophical point of view, this is the whole point of the canonical class!)

2 Applications

2.1 The affine case

Consider first the geometry of the affine graph: the morphism $\pi\colon Y \to X$ is the graph of s. The locus $D \setminus N$, where s has a pole, disappears off "to infinity" on Y, whereas $N \setminus D$ becomes the principal divisor $s = 0$. The intersection $D \cap N$ is the locus of indeterminacy of s, and Y contains an affine line bundle over it.

2.2 Example: nodal curve

The example $X =$ nodal curve, $D =$ reduced node is very instructive; set $X : (x^2 - y^2 = 0)$ and $D : (x = y = 0)$. Then $s = x/y$ is an automorphism of $I = J = \mathfrak{m}$, and $Y \to X$ is an affine blowup, with an exceptional $\mathbb{A}^1$ over the node. (The affine line in question is the projectivised of the 2-dimensional vector space $\mathrm{Hom}_{\mathcal{O}_X}(\mathfrak{m}, \mathfrak{m}) \otimes k$, with the identity element deleted.) This example shows that Remark 1.3, (4) does not hold without the assumption that D and N are Weil divisors.

2.3 Simplest example

We discuss a case that has many consequences in birational geometry, even though the algebra itself is very simple. Consider the generic equations

$$X : (Bx - Ay = 0) \quad \text{and} \quad D : (x = y = 0) \tag{2.1}$$

defining a hypersurface X containing a codimension 2 complete intersection D in some as yet unspecified ambient space. The unprojection variable is

$$s = \frac{A}{x} = \frac{B}{y}\,. \tag{2.2}$$

We can view s as a rational function on X, or as an isomorphism from (x, y) to (A, B) in $\mathcal{O}_X$. The unprojection is the codimension two complete intersection $Y : (sx = A, sy = B)$.

For example, take $\mathbb{P}^3$ as ambient space, with x, y linear forms defining a line D, and A, B general quadratic forms. See 2.4 for the standard trick to make our local construction work also in the projective set-up. Then s has degree 1 from (2.2), and the equations describe the contraction of a line on a nonsingular cubic surface to the point $P_s = (0 : 0 : 0 : 0 : 1) \in \mathbb{P}^4$ on a del Pezzo surface of degree 4. It is the inverse of the linear projection $Y \dashrightarrow X$

from P_s, eliminating s. But the equations are of course much more general. The only assumptions are that x, y and $Bx - Ay$ are regular sequences in the ambient space. For example, if A, B vanish along D, so that X is singular there, then Y contains the plane $x = y = 0$ as an exceptional component lying over D (as in 2.2). Note that, in any case, Y has codimension 2 and is nonsingular at P.

The same rather trivial algebra lies behind the quadratic involutions of Fano 3-folds constructed in Corti, Pukhlikov and Reid [CPR], 4.4–4.9. For example, consider the general weighted hypersurface of degree 5

$$X_5 : (x_0y^2 + a_3y + b_5 = 0) \subset \mathbb{P}(1, 1, 1, 1, 2),$$

with coordinates $x_0, \dots, x_3, y$. The coordinate point $P_y = (0 : \cdots : 1)$ is a Veronese cone singularity $\frac{1}{2}(1, 1, 1)$. The anticanonical model of the blowup of P_y is obtained by eliminating y and adjoining $z = x_0y$ instead, thus passing to the hypersurface

$$Z_6 : (z^2 + a_3z + x_0b_5 = 0) \subset \mathbb{P}(1, 1, 1, 1, 3).$$

The 3-fold Z_6 contains the plane $x_0 = z = 0$, the exceptional $\mathbb{P}^2$ of the blowup. Writing its equation as $z(z + a_3) + x_0b_5$ gives $y = \frac{z}{x_0} = -\frac{b_5}{z+a_3}$, and puts the birational relation between X_5 and Z_6 into the generic form (2.1–2.2). In fact Z_6 is the "midpoint" of the construction of the birational involution of X_5. The construction continues by setting $y' = \frac{z+a_3}{x_0} = -\frac{b_5}{z}$, thus unprojecting a different plane $x_0 = z + a_3 = 0$. For details, consult [CPR], 4.4–4.9. See Corti and Mella [CM] for a related use of the same algebra, to somewhat surprising effect; these and many further examples are treated at more length in Papadakis [P]–[P1] and Reid [R3].

2.4 Projective case

The cases of 2.3 are typical of our applications of unprojection to biregular and birational geometry. Although we developed the theory for local rings in Section 1, it applies at once to projective varieties (schemes) via the standard philosophy of Zariski and Serre summarised in the slogan "graded is a particular case of local" (the coherent half of Grothendieck's "Lefschetz principle", see Grothendieck [G], esp. Chapters I–V). We sketch briefly what we need.

Our graded rings R are graded in positive degrees, with $R_0 = k$ a field, and R finitely generated over k. The associated local ring $R_\mathfrak{m}$ is R localised at the maximal ideal $\mathfrak{m} = \bigoplus_{n>0} R_n$. The principle says that coherent cohomology of sheaves on X can be treated in terms of local cohomology

$H^i_{\mathfrak{m}}(M)$ of modules over $R_{\mathfrak{m}}$ at $\mathfrak{m}$. In particular, Cohen–Macaulay and Gorenstein have equivalent treatments in terms of the geometry of the projective scheme X or the local cohomology of $R_{\mathfrak{m}}$. Geometrically, $\operatorname{Spec} R$ is the affine cone over $X = \operatorname{Proj} R$, and we localise at the origin $(0) = V(\mathfrak{m})$; this replaces the projective variety by a small neighbourhood of the vertex of its affine cone (together with its grading).

Let $I \subset R$ be a graded ideal of codimension 1. We suppose that R and R/I are Gorenstein and write $X = \operatorname{Proj} R$ and $D = \operatorname{Proj} R/I$. Then $D \subset X$ are projectively Gorenstein schemes. Write $\omega_X = \mathcal{O}_X(k_X)$ and $\omega_D = \mathcal{O}_D(k_D)$, and assume that $k_X > k_D$ (see Remark 2.5). The construction of Section 1 gives a rational section s of $\mathcal{O}_X(k_X - k_D)$ that defines an isomorphism $I \to J$ of ideals of R, but with a Serre twist by $k_X - k_D$. (For example, in 2.3, $s\colon (x, y) \mapsto (A, B)$ has degree 1.) It is naturally an isomorphism of sheaves $s\colon I \cong J(k_X - k_D)$. As before, write $R[s] := R[S]/(Sf_i - g_i)$ for the unprojection ring and $Y = \operatorname{Proj} R[s]$.

If $R = k[x_1, \dots, x_k]/(I_X)$ is generated by elements x_i with $\deg x_i = a_i$ then $R[s] = k[x_1, \dots, x_k, s]/(I_Y)$ is generated by x_i together with s, of degree $\deg s = k_X - k_D$. The projective scheme Y contains the distinguished point $P_s = (0 : \cdots : 0 : 1)$. If X is variety (that is, reduced and irreducible), and D a Weil divisor, then $D \cap N$ does not contain any prime divisors, so that the inclusion $R \subset R[s]$ defines a birational map or *unprojection* $X \dashrightarrow Y$ contracting D to the point P_s. This is the striking difference from the affine case, where $D \setminus N$ disappeared "off to infinity". The inverse rational map $Y \dashrightarrow X$ is the projection from P_s, and corresponds to eliminating s. It blows up P_s to a divisor $E \subset \widetilde{Y}$, then defines a morphism $\widetilde{Y} \to Y$ taking E birationally to D.

Remark 2.5 We need the assumption $\deg s = k_X - k_D > 0$ in order that the unprojection ring is still graded in positive degrees. It is interesting to note that it also has the effect of making D have "negative self-intersection". If D is a Cartier divisor on X then of course $\mathcal{O}_D(-D) = \mathcal{O}_D(k_X - k_D)$. However, even if $D \subset \operatorname{Sing} X$, so that it is not even a Weil divisor, the difference $(\omega_X)_{|D} \otimes \omega_D^{-1}$ provides a usable notion of self-intersection class of D. For more discussion of this idea, compare Catanese, Franciosi, Hulek and Reid [CFHR], 3.1–3.2 and [R3], Section 5.3.

Remark 2.6 The case $\dim X = 1$ leads to an apparent paradox, kindly brought to our attention by Nikos Tziolas: Theorem 1.5 applied to a point on a canonical curve $P \in C \subset \mathbb{P}^{g-1}$ seems at first sight to give a canonical

form with a single pole at P. The source of the misunderstanding is that if $\dim D = 0$ then D (up to isomorphism) does not determine k_D; the solution is to keep track of the graded ring over D, which does. In fact $P = \operatorname{Proj} k[x]$, with $\deg x = 1$, so that $k_P = -1$, and $k_C - k_P = 2$. Our construction in this case unprojects C to a curve $C \subset \mathbb{P}(1^g, 2)$. This curve is projectively Gorenstein, with $\mathcal{O}_C(1)$ corresponding to the $\mathbb{Q}$-Cartier divisor $K_C + \frac{1}{2}P$. It is a nonsingular curve passing through the $\mathbb{Z}/2$ quotient singular $P_s = (0, \dots, 0, 1)$, and should be viewed as an orbifold at that point, with orbifold canonical class $K_C + \frac{1}{2}P$.

Exercise 2.7 The humane treatment of the combinatorics of the graded ring $R(X, \mathcal{O}_X(1))$ is in terms of its Poincaré series $P_X(t) = \sum_{n \ge 0} P_n(X) t^n$, where $P_n(X) = h^0(X, \mathcal{O}_X(n))$ (see for example Altınok [A1]). If X, D and Y are as above, prove that

$$P_Y(t) = P_X(t) + \frac{t^k}{1 - t^k} P_D(t), \quad \text{where } k = \deg s = k_X - k_D.$$

2.8 Cases already in the literature

[KM] and [KM1] contain many examples of unprojections in commutative algebra. The following case (already in [KM0]) is probably the simplest: let $A = a_{ij}$ be a generic 3×4 matrix and $x = (x_1, \dots, x_4)$ a 4×1 column vector. Write $q_i = \sum a_{ij} x_j$; these are 3 generic linear combinations of a regular sequence of length 4. Then $D : (x_j = 0)$ is a codimension 4 complete intersection, and $X : (q_i = 0)$ a codimension 3 complete intersection containing D. The unprojection of D in X consists of introducing a new indeterminate s and writing out the equations

$$Ax = 0, \quad sx = \bigwedge^3 A. \tag{2.3}$$

That is, sx_i equals $\pm$ the 3×3 minor $\det A_i$ obtained by deleting the ith row of A. These equations are of course familiar from Cramer's rule. In this case, the complexes $L_\bullet$ and $M_\bullet$ of (1.5) are Koszul complexes, and the vertical arrows are successive wedges of A.

Equations (2.3) give one of the simplest formats of Gorenstein rings in codimension 4, with a 7×12 resolution (7 relations, 12 syzygies). For example: $X \subset \mathbb{P}^6$ a K3 surface given as a complete intersection of 3 quadrics, containing a line D. The unprojection contracts D to an ordinary node of a K3 surface $Y \subset \mathbb{P}(1^7, 2)$ with ideal defined by (2.3).

It is quite curious that the format (2.3) occurs very rarely in nature, possibly because it does not have a grading with all generators of degree 1. For example, the canonical ring $R(X, K_X)$ of a regular surface of general type with $p_g = 5, K^2 = 11$ has Hilbert series $P_X(t)$ given by

$$\begin{aligned} P_X(t) &= \frac{1 + (p_g - 3)t + (K^2 - 2p_g + 4)t^2 + (p_g - 3)t^3 + t^4}{(1-t)^3} \\ &= \frac{1 - 6t^3 - t^4 + 12t^5 - t^6 - 6t^7 - t^{10}}{(1-t)^5(1-t^2)^2}, \end{aligned}$$

so a first hope is that $X \subset \mathbb{P}(1^5, 2^2)$ could be defined by 6 relations in degree 3, and 1 relation in degree 4, yoked by 12 syzygies in degree 5. In fact, for geometric reasons, the two generators y_1, y_2 of degree 2 must occur in 3 degree 4 relations $y_1^2 = \cdots$, etc., so that the resolution must be at least 9×16.

In contrast, rings with 9×16 resolutions are ubiquitous, and we know a couple of hundred cases. Most of the families of K3s and Fano 3-folds in codimension 4 studied in Altınok [A] (more than a hundred of them; see also [R2]) are unprojections in weighted projective space. Their rings almost invariably have 9×16 resolutions. There is a vaguely formulated (but strongly documented) conjecture that rings with 9×16 resolutions fall into 2 standard families called Tom and Jerry, bearing a family resemblance to the Segre embeddings of $\mathbb{P}^2 \times \mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. See [P] and [P1] for details of the families as unprojections and compare [R3], Section 8.

2.9 Gorenstein projections and discrepancy

When we project a del Pezzo surface Y of degree $d \geq 2$ to another X of degree $d-1$, the original surface satisfies $K_Y = -H$, the blowup $\widetilde{Y} \to Y$ introduces a -1-curve E of discrepancy 1, so that $K_{\widetilde{Y}} = -\widetilde{H} + E$, and the projection morphism $\widetilde{Y} \to X$ is given by $\widetilde{H} - E = -K_{\widetilde{Y}}$. Thus the condition that both X and Y are projectively Gorenstein (here, anticanonically polarised) involves an implicit discrepancy calculation. We avoid a formal definition of Gorenstein projection (except in the simplest case given below), because the notion makes sense at different levels of generality. We only explain briefly the point about discrepancy for the reader versed in 3-folds.

Consider a projectively Gorenstein variety Y, H with $K_Y = kH$, a point blowup $\sigma\colon \widetilde{Y} \to Y$ extracting a divisor E with discrepancy a, with a new divisor $\widetilde{H} = H - mE$ that defines a small contraction $\varphi\colon \widetilde{Y} \to X$ with

$\varphi(E) = D \subset X$. Suppose that also $K_X = kH_X$; then because $\widetilde{Y} \to X$ is small, also $K_{\widetilde{Y}} = k\widetilde{H}$, and

$$K_{\widetilde{Y}} = K_Y + aE, \quad \widetilde{H} = H - mE \implies a + km = 0. \tag{2.4}$$

For Fano 3-folds, $k = -1$, and the basic operation in [CPR] is the Kawamata blowup of a quotient singularity $\frac{1}{r}(1, a, r-a)$, which satisfies $a = m = \frac{1}{r}$. For Y a K3 surface with Du Val singularities, $k = 0$, and any crepant blowup (that is, with $a = 0$) may lead to a Gorenstein projection (and does so provided that $H - mE$ is still nef and big). Another interesting case is a canonically embedded regular surface Y with (say) a simple elliptic singularity. Then $k = 1$, $a = -1$, and the linear projection may be a Gorenstein projection; this type of blowup decreases $p_a Y$ by 1, and K_Y^2 by the degree of the singularity, and appears in various forms in many constructions of algebraic surfaces of general type.

We give a temporary definition to serve as a converse to Theorem 1.5. Say that $Y \dashrightarrow X$ is a *Gorenstein projection of Type I* if $\sigma\colon \widetilde{Y} \to Y$ is a point extraction with exceptional divisor E, and $\varphi\colon \widetilde{Y} \to X$ a small birational morphism taking E birationally to $D \subset X$, such that X and D are both projectively Gorenstein. In this case X, D satisfies the assumptions of Theorem 1.5, and the unprojection leads back to Y. Necessary conditions for this to hold are the restriction (2.4) on the discrepancy of the blowup, and the surjectivity of the restriction maps $H^0(\widetilde{Y}, n\widetilde{H}) \to H^0(E, n\widetilde{H}_{|E})$ (needed so that D is projectively normal).

2.10 More general D

Finally, our construction has natural generalisations: the exceptional divisor $D \subset X$ of a Gorenstein projection is not restricted to being projectively Gorenstein. For example, Fano's construction of his 3-folds $V_{2g-2} \subset \mathbb{P}^{g+1}$ for $g \geq 6$ involves projections $Y \dashrightarrow X$ from a line, giving rise to the exceptional locus $D \subset X$ a cubic scroll. Alessio Corti observed that the inverse map can be treated by a generalisation of our method: to unproject D, we need to take two generators of $\mathcal{H}om(I_D, \omega_X)$ of degree 1, that map down to a basis of $\omega_D(2)$, the ruling of the scroll. (He also carried out in some detail the algebraic calculation of the anticanonical ring of Y in these terms.)

In the weighted projective case, it can happen that $\omega_D \cong \mathcal{O}_D(k_D)$, but D is not projectively normal. For example, consider the following method for constructing Fano 3-folds: let D be a weighted projective plane $\mathbb{P}(1, b_1, b_2)$ (for some small values of b_1, b_2), embedded as a projectively nonnormal

surface in some $\mathbb{P}^4(1, a_1, a_2, a_3, a_4)$. A Fano hypersurface X_d with mild singularities can contain D, and then D can be contracted in X to a terminal quotient singularity $\frac{1}{b_1+b_2}(1, b_1, b_2)$ in a Fano 3-fold Y by an unprojection.

Thus D can sometimes be unprojected even though it is not Gorenstein, for example because it is not normal. Arguing locally as in Section 1, suppose that D is a divisor in a normal Gorenstein variety X, that D is nonnormal, but its normalisation map $\widetilde{D} \to D$ is an isomorphism in codimension 1, and such that $\widetilde{D}$ is Gorenstein. For example, D could be $\mathbb{A}^2$ with the subscheme $(x^2 = y = 0)$ pinched to a point, or $\mathbb{A}^3$ with the x-axis glued to itself by $x \mapsto -x$. Since $\omega_D = \omega_{\widetilde{D}}$, it is generated by a single element $\overline{s}$ as a module over the normalisation $\mathcal{O}_{\widetilde{D}}$. Thus $\mathcal{H}om(I_D, \omega_X)$ contains a generator s mapping to the basis $\overline{s} \in \omega_{\widetilde{D}}$. From the birational point of view, s has a pole along D, so its graph sends D off to a point at infinity. Thus adjoining s unprojects D in a variety Y_0. We believe that the normalisation $Y \to Y_0$ is a Gorenstein variety (see [R3], Problem 9.2). In this case, the algebraic presentation of the unprojection ring $\mathcal{O}_Y$ is of course much more complicated, and only worked out in special cases. The normalisation amounts to taking a few extra generators $s_0 = s, s_1, \dots, s_k \in \mathcal{H}om(I_D, \omega_X)$, corresponding to generators of ω_D over $\mathcal{O}_D$ (rather than $\mathcal{O}_{\widetilde{D}}$), and adjoining these to $\mathcal{O}_X$. The case we understand is discussed in [CPR], 7.3 and [R3], Section 9; see also the corresponding discussion in [CM] and [R4]. Although the unprojection ring $\mathcal{O}_X[s_0, \dots, s_k]$ in this case certainly has relations of the form $s_i s_j = \cdots$ that are quadratic in the s_i, it is still uniquely determined by its linear relations, essentially coming from the presentation of ω_D over $\mathcal{O}_D$.

References

[A] S. Altınok, Graded rings corresponding to polarised K3 surfaces and $\mathbb{Q}$-Fano 3-folds, Univ. of Warwick Ph.D. thesis, Sep. 1998, 93 + vii pp., get from www.maths.warwick.ac.uk/~miles/doctors/Selma

[A1] S. Altınok, Hilbert series and applications to graded rings, submitted

[BE] David A. Buchsbaum and David Eisenbud, What makes a complex exact? J. Algebra **25** (1973) 259–268

[BH] W. Bruns and J. Herzog, Cohen–Macaulay rings, CUP 1993

[CFHR] F. Catanese, M. Franciosi, K. Hulek and M. Reid, Embeddings of curves and surfaces, Nagoya Math. J. **154** (1999) 185–220

[CM] A. Corti and M. Mella, Birational geometry of terminal quartic 3-folds. I, preprint math.AG/0102096, 37 pp.

[CPR] A. Corti, A. Pukhlikov and M. Reid, Birationally rigid Fano hypersurfaces, in Explicit birational geometry of 3-folds, A. Corti and M. Reid (eds.), CUP 2000, 175–258

[G] Alexander Grothendieck, Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA_2). North-Holland and Masson & Cie, 1968

[KM0] A. Kustin and M. Miller, Algebra structures on minimal resolutions of Gorenstein rings of embedding codimension four, Math. Z. **173** (1980) 171–184

[KM] A. Kustin and M. Miller, Constructing big Gorenstein ideals from small ones, J. Algebra **85** (1983) 303–322

[KM1] A. Kustin and M. Miller, Deformation and linkage of Gorenstein algebras, Trans. Amer. Math. Soc. **284** (1984) 501–534

[M] Hideyuki Matsumura, Commutative ring theory, Second edition, C.U.P., 1989

[P] Stavros Papadakis, Gorenstein rings and Kustin–Miller unprojection, Univ. of Warwick Ph.D. thesis, Aug 2001, vi + 72 pp., get from www.maths.warwick.ac.uk/∼miles/doctors/Stavros

[P1] Stavros Papadakis, Kustin–Miller unprojection *with* complexes, submitted, math.AG/0111195, 23 pp.

[R] M. Reid, Nonnormal del Pezzo surfaces, Math Proc. RIMS **30** (1994) 695–727

[R1] M. Reid, Undergraduate commutative algebra, CUP 1995

[R2] M. Reid, Graded rings over K3 surfaces, in abstracts of Matsumura memorial conference (Nagoya, Aug 1996), 11 pp.

[R3] M. Reid, Graded rings and birational geometry, first draft in Proc. of algebraic geometry symposium (Kinosaki, Oct 2000), K. Ohno (Ed.), 1–72; get from: www.maths.warwick.ac.uk/∼miles/3folds

[R4] M. Reid, Examples of Type IV unprojection, preprint math.AG/0108037, 16 pp.

[WITO] K-i Watanabe, T. Ishikawa, S. Tachibana and K. Otsuka, On tensor products of Gorenstein rings, J. Math. Kyoto Univ. **9** (1969) 413–423

Stavros Papadakis,
Math Inst., Univ. of Warwick,
Coventry CV4 7AL, England
e-mail: spapad@maths.warwick.ac.uk

and Karaiskaki 11, Agia Paraskevi,
GR 153 41, Attiki, Greece
e-mail: stavrospapadakis@hotmail.com

Miles Reid,
Math Inst., Univ. of Warwick,
Coventry CV4 7AL, England
e-mail: miles@maths.warwick.ac.uk
web: www.maths.warwick.ac.uk/∼miles